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BARGAINING OVER RISKY ASSETS

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Bargaining over Risky Assets[1]

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Abstract

We analyze the subgame-perfect equilibria of a game where two agents bargain in order to share the risk in their assets that will pay dividends once at some fixed date. The uncertainty about the size of the dividends is resolved gradually by the payment date and each agent has his own view about how the uncertainty will be resolved. As agents become less uncertain about the dividends, some contracts become unacceptable to some party to such an extent that at the payment date no trade is possible. The set of contracts is assumed to be rich enough to generate all the Pareto-optimal allocations. We show that there exists a unique equilibrium allocation, and it is Pareto-optimal. Immediate agreement is always an equilibrium outcome; under certain conditions, we further show that in equilibrium there cannot be a delay. Finally, we characterize the conditions under which every Pareto-optimal and individually rational allocation is obtainable via some bargaining procedure as the unique equilibrium outcome.

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1 Introduction

Consider two risk-averse agents who want to share the risk in their assets. Assets will pay dividends only once, at some fixed date $\bar{t}$. As in Wilson (1968), each agent has his own beliefs about the dividends, but no agent has any private information.[1] Which contract will these agents agree on, and when will they reach an agreement? What are the determinants of "bargaining power"? These are the questions we ask in this paper. One might find it obvious that, given a sufficiently rich set of contracts, the agents will immediately agree on a Pareto optimal contract. We will show that this is true – but there are some interesting details to it.

In this environment, any allocation feasible at the beginning remains feasible until the end, and the agents are indifferent about which date they agree on any *given* allocation. Thus, the set of payoffs does not shrink as time passes and there is no discounting to create impatience for agreement. Instead, as time passes, some uncertainty about the dividends is resolved, making some of the contracts individually irrational for some agent, given the agent's expectations about how events will proceed. This reduces the scope for insurance. (Eventually at date $\bar{t}$ agents lose all insurance opportunities; autarky is the only individually rational contract left.) This loss of contracting opportunities leads agents to agree early on.

One can imagine many bargaining problems where such *lost contracting opportunities are the major costs of delay* – not discounting by the agents or some material losses. For example, many wage contracts are negotiated before the current contract expires. Thus, until the current contract expires, a major part of the cost of delay is the lost insurance opportunities between the workers and the firm against fluctuations in the labor market.[2] Sometimes, agents may also hold incompatible beliefs about the size of the gains from trade and each agent's outside options. Our analysis here may be helpful in analyzing such problems.

Our efficiency and immediate-agreement results are as follows. Assuming that all Pareto-optimal allocations can be generated by contracts throughout the bargaining process, under any sequential bargaining procedure with immediately expiring offers, the following are true: There exists an equilibrium where agents reach an agreement immediately; any subgame-perfect equilibrium is payoff-equivalent to this equilibrium (Theorem 1); and the equilibrium allocation is unique and Pareto-optimal when the first offer is made (Theorem 2).[3] Furthermore, except

[1] Under these conditions, assuming that a Pareto-optimal allocation rule is employed, Wilson (1968) characterizes the conditions under which the group can be represented by an expected-utility maximizer.

[2] Another example: it is common in legal practice that the settlement is negotiated while parts of the case are litigated. As each decision is made in court, some uncertainty about the eventual payment by the defendant is resolved, making certain settlement agreements individually irrational for certain parties. (In wage bargaining, delaying agreement may also result in the costly delay of some decisions the parties might want to take after the agreement. In litigation, the legal fees are also very large.)

[3] The proof of Theorem 1 is somewhat more involved than the argument that, given a sufficiently rich set of contracts, the agents can imitate any delay by writing a contingent contract that requires at each contingency

for the trivial case where autarky is initially Pareto-optimal, immediate agreement is the only equilibrium outcome, provided each agent can make an offer with positive probability throughout (Theorem 3).

These results clarify certain results in the bargaining literature. Excessive optimism is presented as an explanation for delays by many authors, such as Hicks (1932), Landes (1971), Posner (1974), Farber and Bazerman (1989), and Yildiz (2001). They demonstrate that, if each agent has excessively high expectations about his prospects in case of a delay, then delay may be necessary in equilibrium. Here, under the condition that agents can generate all the Pareto-optimal allocations throughout, we show that immediate agreement is always an equilibrium outcome, and is in fact the only equilibrium outcome for generic cases. Therefore, the delay results from the authors' restrictions on the set of contracts. Moreover, in order to satisfy our condition, all we need is risk-sharing agreements about the dividends and side-bets on the events about which our agents have distinct beliefs. Therefore, their results are based particularly on their implicit assumption that side-bets are not feasible.

We show that the continuation value of an agent at the beginning of any given date is the sum of the rents he expects to extract when he makes offers in the future, plus his expected utility of consuming the dividend paid by his own asset. The value of the rents is determined by the way uncertainty is resolved. After a substantial resolution of uncertainty, a substantial group of the potential contracts become individually irrational, and thus not viable. Therefore, agents who make offers prior to such resolution of uncertainty extract substantial rents. For instance, in the canonical case analyzed in Section 5, revealing an informative signal at some time t rather than s is equivalent to transferring the gain from trade associated with this signal from the agent who makes an offer at $s-1$ to the agent who makes an offer at $t-1$.

Many axiomatic bargaining solutions have the property that, if an agent becomes more risk-averse, then he becomes worse off while his opponent becomes better off. The same property holds in the Rubinstein-Stahl model for the *instantaneous* risk-aversion. In our model, the impact of risk aversion is ambiguous. If the total wealth in society is risky, risk aversion of an agent may hurt his opponent. Hence, this property exhibited (and sometimes postulated) in axiomatic bargaining models is not confirmed here.

Finally, even though we allow our agents to hold different beliefs, this is not the main attribute of our model. In fact, the earlier versions of this paper assumed that the agents hold the same beliefs.

The outline of the paper is as follows. In the next section we lay out our model. In Section 3, we present examples with delay and inefficiency. We present our main results in Section 4. In Section 5, we derive the solution in closed form for a canonical case; we illustrate our results,

that they write the very contract they would have written in the case of delay. This argument implicitly assumes that there exists an essentially unique equilibrium in the continuation game, a result that needs a proof.

and discuss the effects of risk aversion. In Section 6, we present a monotonicity property of the equilibrium and characterize the allocations that can be the equilibrium outcome for some bargaining procedure. The appendix contains the proofs omitted in the text.

2 Model

In this section we lay out our model. For some (possibly large) $\bar{t} \in \mathbb{N}$, we take $T = \{t \in \mathbb{N} | 0 \leq t \leq \bar{t}\}$ to be the time space, where $\mathbb{N}$ denotes the set of all non-negative integers. At each date $t \in T$, a signal Y_t becomes publicly observable and remains observable thereafter. Writing $Y = (Y_0, \ldots, Y_{\bar{t}})$ and $Y^t = (Y_0, \ldots, Y_t)$, we designate $y \equiv \{y_s\}_{s \in T}$ and $y^t \equiv \{y_s\}_{s \leq t}$ as the generic realizations of Y and Y^t, respectively.

We have one consumption good, which will be consumed at date $\bar{t}$, and two agents, whose set will be denoted by $N = \{1, 2\}$. Each agent i has a strictly concave and strictly increasing Von-Neumann-Morgenstern utility function $u^i : \mathbb{R} \to \mathbb{R}$. We write $P^i \left(\cdot | y^t\right)$ for the beliefs of i at y^t, and P^i for his prior beliefs. We assume that P^1 and P^2 are equivalent. That is, they assign zero probability to the same events. We write $E^i \left[\cdot | y^t\right]$ and E^i for the expectations with respect to $P^i \left(\cdot | y^t\right)$ and P^i, respectively.

Given any $i \in N$, for some $\underline{x}^i \in \mathbb{R}$, we take $[\underline{x}^i, \infty)$ as the set of feasible consumption levels for agent i and write X^i for the set of all (risky) consumption functions, i.e., the random variables $x^i : y \mapsto x^i(y) \in [\underline{x}^i, \infty)$ that are integrable with respect to P^i. Each agent $i \in N$ has an asset that pays only one dividend $D^i \in X^i$, which will be paid at $\bar{t}$. There is no market at which our agents can trade.

By an *allocation*, we mean any $x \in X^1 \times X^2$ with $x^1(y) + x^2(y) \leq D^1(y) + D^2(y)$ at each y. Here, $x^i(y)$ denotes the consumption of agent i when the realization of Y is y. By a *contract* $x[y^t]$ *at* a given y^t, we simply mean the restriction of an allocation x to the set $\{y' | y'_s = y_s \text{ for all } s \leq t\}$ of all continuations of y^t. Clearly, an allocation x is the mutual extension of contracts $x[y^t]$. We write $X\left[y^t\right]$ for the set of all contracts at y^t and X for the set of all allocations. We take some $C_t\left[y^t\right] \subseteq X\left[y^t\right]$ as the set of feasible contracts at y^t and write $C_t \subseteq X$ for the set of allocations generated by the contracts in $C_t\left[y^t\right]$. (The set of feasible contracts may be strictly smaller due to some additional constraints.) We finally write $C = (C_0, \ldots, C_{\bar{t}})$.

We will analyze the game $G(C, \rho)$ in which the following is common knowledge: At each t, first the signal Y_t is observed and an agent $\rho_t\left(y^t\right) \in N$ is recognized. Then, the recognized agent offers a contract $x\left[y^t\right]$. If the contract is accepted by the other agent, they sign the contract, yielding payoff $E^i\left[u^i\left(x\left[y^t\right]\right) | y^t\right]$ for each i; otherwise, the offer expires and they wait until the next date, except for date $\bar{t}$, when the game ends and each i consumes $D^i(y)$. The process $\rho \equiv (\rho_0, \ldots, \rho_{\bar{t}})$ is called the *recognition process*.

We assume[4] that $P^1(\rho_t = i|y^s) = P^2(\rho_t = i|y^s) = p_t^i(y^s)$ for some $p_t^i(y^s)$ at each t and y^s. We call any full list $p = \{p_t^i(y^s)\}_{y^s,i,t}$ of such probability assessments a *bargaining procedure*. We assume that ρ and D are independent and that $p_t^i(y^s)$ depends only on the history $\{\rho_{t'}(y_0, \ldots, y_{t'})\}_{t' \leq s}$ of recognized agents. We finally assume that ρ is affiliated: when an agent is recognized at a given date, he will be, if anything, more likely to be recognized in the future. The set of all such bargaining procedures is denoted by $\mathcal{P}$.

Optimality, side bets, risk-sharing agreements Given any y^t, an allocation $\bar{x}$ is said to be *Pareto-optimal at* y^t iff

$$\left[\exists i \in N,\ E^i[u^i(x)|y^t] > E^i[u^i(\bar{x})|y^t]\right] \Rightarrow \left[\exists j \in N,\ E^j[u^j(\bar{x})|y^t] > E^j[u^j(x)|y^t]\right],$$

almost surely, for each allocation $x \in X$. That is, the (conditional) probability that we can improve both agents' expected utility at y^t is zero. An allocation is said to be *Pareto-optimal* iff it is Pareto-optimal at the beginning. We say that C is *Pareto-complete* iff C_t contains all Pareto-optimal allocations for each $t \in T$.

Our next lemma states that C is Pareto-complete whenever all risk-sharing agreements and side-bets are available. Let us consider some objective components of uncertainty separately, and write each Y_t as $(\tilde{Y}_t, \bar{Y}_t)$ where the agents have a common prior on $\tilde{Y}_t$, which contains information about some objective aspects of uncertainty. Likewise, we write each y as $(\tilde{y}, \bar{y})$.

Lemma 1 *Let* $\bar{X} = \{x \in X | x(\tilde{y}, \bar{y}) = x(\tilde{y}', \bar{y})\ \forall (\tilde{y}, \tilde{y}', \bar{y})\}$. *If* $D \in \bar{X}$, *then* $(\bar{X}, \ldots, \bar{X})$ *is Pareto-complete.*

That is, Pareto-optimal contracts do not refer to $\tilde{Y} = (\tilde{Y}_0, \ldots, \tilde{Y}_{\bar{t}})$ if $\tilde{Y}$ does not affect the dividends and if agents hold common beliefs about $\tilde{Y}$. Put differently, so long as we can write contracts on all the events that are relevant for the dividends (so that they can share all the risk) or the events about which the agents have distinct beliefs (so that they can bet), we can generate all the Pareto-optimal contracts.

3 Examples of delay and inefficiency

Two properties of the bargaining procedures in this paper are essential: (i) they are sequential and (ii) the offers expire immediately. In addition, we assume that (iii) C is Pareto-complete. Under these conditions, we will show that all equilibria are equivalent to an equilibrium where players immediately agree on an allocation that is Pareto-optimal when the first offer is made.

[4] All the regularity conditions we are about to assume are here to simplify the exposition. They are only used in Theorems 3 and 4 (see Yildiz, 2000).

(We will further show that this is the only equilibrium outcome for generic cases.) Now, we will demonstrate that these conditions are not superfluous.

Our first example demonstrates that, when C is not Pareto-complete, we may have delay and inefficiency in (the unique) equilibrium. In this example, we restrict our contracts to the *sales contracts.* A sales contract can be represented by the amounts of equity shares Agent 1 owns in the assets, and a consumption-good transfer from Agent 2 to Agent 1.

Example 1 (The set of contracts is Pareto-incomplete.) *Take $T = \{0, 1, 2\}$, $\rho_0 = \rho_2 = 1$, and $\rho_1 = 2$. For each i, take $\underline{x}^i = 0$ and $u^i(x^i) = \sqrt{x^i}$. Take Y_0 a constant, and D (as a function of Y_1 and Y_2) as in the following table.*

	$Y_2 = H$	$Y_2 = L$
$Y_1 = 1$	$(16, 9)$	$(16, 0)$
$Y_1 = 2$	$(9, 16)$	$(0, 16)$

We take $P^i(Y_1 = i) = 1$ and $P^i(Y_2 = H) = P^i(Y_2 = L) = 1/2$ for each i.[5] *We assume that only sales contracts are feasible.*

At date 2, each agent knows the true state, hence any trade will be just a transfer from some agent i to the other, which is not individually rational for i. Hence, if they have not agreed by date 2, they will not trade. Thus, Agent 1 will accept an offer at $t = 1$ iff it gives him at least the expected utility of consuming his own asset. Moreover, at $t = 1$, the optimal contracts are sales contracts where any agent's equity shares in two assets are the same. Hence, at $t = 1$, Agent 2 will offer the Pareto-optimal sales contract that gives Agent 1 his continuation value. The offer will be accepted. When $Y_1 = 1$, the asset of 1 will pay 16 for sure, yielding expected utility of $\sqrt{16} = 4$. Then, Agent 2 will offer 64/81 of each asset to Agent 1, yielding payoff vector $(4, \sqrt{15}/2)$. When $Y_1 = 2$, if Agent 1 consumes his own asset, his expected utility will be only $\sqrt{9}/2 + \sqrt{0}/2 = 3/2$. In that case, Agent 2 will offer 1/9 of each asset to Agent 1, yielding payoff vector $(3/2, 3\sqrt{2})$. At $t = 0$, Agent 1 is certain that $Y_1 = 1$ and he will get 4. Likewise, Agent 2 is certain that $Y_1 = 2$ and she will get $3\sqrt{2}$. But the best sales contract at $t = 0$ is no-trade, yielding expected payoff $(4, 4) \ll (3/2, 3\sqrt{2})$. Therefore, they disagree at $t = 0$.

Yielding payoff vector $(4, 3\sqrt{2})$, the equilibrium is Pareto-inefficient. If Agent 1 owned both assets when $Y_1 = 1$ and Agent 2 owned both assets when $Y_1 = 2$ (the sales contracts available at each $t \in \{1, 2\}$), the payoff vector would be $(9/2, 9/2)$.

Note that there is no allocation generated by feasible contracts at $t = 0$ that Pareto-dominates the equilibrium allocation. Therefore, the inefficiency is *not* caused by delay. Recognizing that the Pareto-dominating contracts will be individually irrational when they

[5] For simplicity, we take $P^i(Y_1 = i) = 1$ for each i, when P^1 and P^2 will not be equivalent. For sufficiently small $\epsilon > 0$, $P^i(Y_1 = i) = 1 - \epsilon$ would also work.

become available, agents agree on a Pareto-dominated contract. Note also that, if C were Pareto-complete, the contract that gives the entire wealth to the agent i with $Y_1 = i$ would be feasible at $t = 0$; and agents would agree on this contract at $t = 0$.

When the bargaining procedure is not sequential, there may be multiple equilibria. In that case, the equilibria in future subgames may depend on the play at the beginning; and this may cause a disagreement at the beginning. This is demonstrated in our next example. (Whenever a subgame possesses multiple equilibria, none of which dominates the other, we can add a prior date and construct a subgame-perfect equilibrium of the new game with delay.)

Example 2 *(*The bargaining procedure is non-sequential.*) Take $T = \{0, 1, 2\}$, $\underline{x}^i = 0$, and $u^i(x^i) = \sqrt{x^i}$ for each i. Let Y_0 be a constant, Y_1 and Y_2 be identically and independently distributed, each taking values of 0 and 1 with equal probabilities. Take $D = 4(Y_1, 1 - Y_1)$ so that the social wealth $D^1 + D^2 = 4$ is riskless. At dates 0 and 2, we have sequential bargaining procedures with $\rho_0 = \rho_2 = 1$. At $t = 1$, simultaneously, each agent offers a feasible contract. If they offer the same contract, it becomes enforceable; otherwise, they wait until date 2.*

If they have not agreed by date 2, they do not trade, and each gets $\sqrt{4}/2 + \sqrt{0}/2 = 1$. There are multiple equilibrium substrategies at $t = 1$. In one of them, agents agree on the riskless allocation $(1, 3)$, where agents 1 and 2 consume 1 and 3 (for sure), respectively. In another one, they agree on the riskless allocation $(3, 1)$. One can check that the following is an equilibrium. If $Y_1 = 0$ or Agent 1 offers $(4, 0)$ at $t = 0$, agents agree on the riskless allocation $(3, 1)$ at $t = 1$; they agree on $(1, 3)$ otherwise. At $t = 0$, Agent 2 accepts an offer x iff $E\left[\sqrt{x^2}\right] \geq \sqrt{3}$; and Agent 1 offers $(4, 0)$, which is rejected. [This is an inefficient equilibrium; it selects $(1, 3)$ and $(3, 1)$ with equal probabilities.]

Finally, our next example demonstrates that, if the offers lived longer, the equilibrium outcome might necessarily be inefficient.

Example 3 *(*The offers live longer.*) Take $T = \{0, 1\}$, and assume that the offers made at $t = 0$ expire at the end of date 1. Let $\rho_0 = 1$ so that Agent 1 makes the offer at 0. Given any offer $x \neq D$ of Agent 1, the best response of Agent 2 is to wait until date $t = 1$ and to accept the offer iff $x^2(y) \geq D^2(y)$. Hence, for any offer x, Agent 1 would consume $\min\{x^1, D^1\}$ and Agent 2 would consume $\max\{x^2, D^2\}$. In that case, Agent 1 would offer $x = D$. Therefore, there is no trade in equilibrium.*

Similarly, allowing the offers to live one period in Rubinstein's (1982) model, Avery and Zemsky (1994) show that the steady-state subgame-perfect equilibrium may be inefficient. In many situations some information may arrive before the offer expires. This adds an option value for waiting, and causes inefficiency. (In an earlier version of this paper, we show that, if C is Pareto-complete, and the offers live one period, the equilibrium allocation will be Pareto-optimal *when* the first offer expires, that is, conditional on the signal observed.)

4 Equilibrium

In this section we analyze the subgame-perfect equilibria of a game $G(C, \rho)$ with Pareto-complete C. We first construct an equilibrium where agents reach an agreement immediately. All the equilibria yield the same equilibrium allocation, which is Pareto-optimal when the first offer is made. We show further that immediate agreement is the only equilibrium outcome if there are some gains from trade and each agent is likely to make an offer throughout. Finally, we derive an equation determining an agent's equilibrium payoff in terms of the dividends and the sequence of expected rents.

As usual, a strategy of an agent is a complete contingent-plan that determines the offer the agent makes when he makes an offer, and whether he accepts or rejects an offer made by the other agent. We require an agent's offers at various y^t of a given date t to be measurable. A strategy profile $\mathrm{s} = (\mathrm{s}^1, \mathrm{s}^2)$ is a subgame-perfect equilibrium (henceforth simply equilibrium) iff the strategies are best response to each other in each subgame.

We first present two preliminary results used in our construction. Our first Lemma states some familiar properties of optimization with increasing functions.

Lemma 2 *Let C be Pareto-complete. Given any distinct $i, j \in N$, any t, any y^t, and any $v \equiv \left(E^1\left[u^1\left(z^1\right)|y^t\right], E^2\left[u^2\left(z^2\right)|y^t\right]\right)$ with $z = (z^1, z^2) \in X$, the optimization problem*

$$\max_{x[y^t] \in C_t[y^t]} E^i[u^i(x^i[y^t])|y^t] \quad \text{subject to} \quad E^j\left[u^j\left(x^j[y^t]\right)|y^t\right] \geq v^j \tag{1}$$

has an (almost surely) unique solution $\hat{x}_t[v, i; y^t]$, which does not depend on C. Write $m^i(v; y^t) = E^i[u^i\left(\hat{x}^i_t[v, i; y^t]\right)|y^t]$ for the maximum. Then, the following are true.

1. *$\hat{x}_t[v, i; y^t]$ is Pareto-optimal at t;*

2. *the constraint is binding, i.e., $E^j\left[u^j\left(\hat{x}^j_t[v, i; y^t]\right)|y^t\right] = v^j$; and*

3. *$m^i\left(v; y^t\right) \geq v^i$, where the inequality is strict whenever z is not Pareto-optimal at y^t.*

Given any two dates $t, s \in T$ with $t \leq s$, consider a collection $\{x[y^s]\}_{y^s}$ of contracts at s, indexed by the continuation y^s of y^t until s. Write $x|y^t$ for the mutual extension of these contracts. At each y^s, $x|y^t$ requires agents to write $x[y^s]$. We call $x|y^t$ *preemptive*, for $x|y^t$ is equivalent to offering the continuation in advance to preempt the delay. Our next Lemma states that $x|y^t$ will be a contract at y^t in all cases of concern.

Lemma 3 *Take any $t \in T$, and let $v\left[y^s\right] \equiv \left(E^1\left[u^1\left(z^1\right)|y^s\right], E^2\left[u^2\left(z^2\right)|y^s\right]\right)$ at each y^s for some $\left(z^1, z^2\right) \in X$. Then, we can select a family $\hat{x}_s[v\left[y^s\right], i; y^s]$ of solutions (indexed by y^s) so that their mutual extension will also be an allocation.*

Construction of an Equilibrium For a given $G(C,\rho)$ with Pareto-complete C, we now construct a subgame-perfect equilibrium, where the agents reach an agreement immediately. We use backward induction. If the agents have not agreed by $\bar{t}$, at $\bar{t}$, agent $j \neq \rho_{\bar{t}}(y)$ will accept an offer $x(y)$ iff $x^j(y) \geq D^j(y)$. This inequality holds only when $x^k(y) \leq D^k(y)$ for $k = \rho_{\bar{t}}(y)$. Hence, k will offer $x_{\bar{t}}(y) = D(y)$. There will be no trade. Each will consume his own asset, independent of which contracts were offered at previous dates. The continuation value of an agent i at the beginning of $\bar{t}$ will be

$$V^i_{\bar{t}}\left(y^{\bar{t}-1}\right) = E^i\left[u^i\left(D^i\right)|y^{\bar{t}-1}\right]. \tag{2}$$

Given any date $t < \bar{t}$, assume that an allocation $x_{t+1} \in X$ will prevail at date $t+1$, independent of which contracts were offered (and rejected) before. Write $V^i_{t+1}(y^t) = E^i\left[u^i\left(x^i_{t+1}\right)|y^t\right]$ for the continuation value of any agent i at the beginning of $t+1$. Given that agent $j \neq \rho_t\left(y^t\right)$ gets $V^j_{t+1}(y^t)$ when he rejects an offer $x_t[y^t]$, it is a best response for him to accept $x_t[y^t]$ iff $E^j[u^j(x^j_t[y^t])|y^t] \geq V^j_{t+1}(y^t)$. Given this response, if agent $k = \rho_t\left(y^t\right)$ offers a contract $x_t[y^t]$, then his expected utility-level will be $E^k[u^k(x^k_t[y^t])|y^t]$ if $E^j[u^j(x^j_t[y^t])|y^t] \geq V^j_{t+1}(y^t)$, and $V^k_{t+1}(y^t)$ otherwise. To maximize this expected utility level, he will either offer $\hat{x}_t[V_{t+1}(y^t),k;y^t]$, the best acceptable offer, and get $m^k\left(V_{t+1}(y^t);y^t\right)$, or will offer some unacceptable offer and get $V^k_t(y^t)$. By Lemma 2.3, $m^k\left(V_{t+1}(y^t);y^t\right) \geq V^k_{t+1}(y^t)$, hence offering $\hat{x}_t[V_{t+1}(y^t),k;y^t]$ is a best response for k. Agent k offers $\hat{x}_t[V_{t+1}(y^t),k;y^t]$, and the offer is accepted. Since $x_{t+1} \in X$, by Lemma 3, we can choose contracts $\hat{x}_t[V_{t+1}(y^t),k;y^t]$ at all sample paths y^t in such a way that their mutual extension forms an allocation, yielding a (preemptive) contract $\hat{x}_t[V_{t+1},k;\cdot]|y^{t-1}$ at y^{t-1}. We choose them so. If agents do not reach an agreement by date t, allocation $\hat{x}_t[V_{t+1},\rho_t;\cdot]$ (which is $\hat{x}_t[V_{t+1},i;\cdot]|y^{t-1}$ when $\rho_t\left(y^t\right) = i$ and $\hat{x}_t[V_{t+1},j;\cdot]|y^{t-1}$ when $\rho_t\left(y^t\right) = j$) prevails at t. This allocation is once again independent of which contracts were offered previously. Since $u^i\left(\hat{x}^i_t[V_{t+1},i;\cdot]\right) = m^i\left(V_{t+1};\cdot\right)$ and $u^i\left(\hat{x}^i_t[V_{t+1},j;\cdot]\right) = V^i_{t+1}(\cdot)$, the continuation value of an agent i at the beginning of t is

$$V^i_t(y^{t-1}) = E^i[m^i\left(V_{t+1};y^t\right)1_{\{\rho_t(y^t)=i\}}|y^{t-1}] + E^i[V^i_{t+1}\left(y^t\right)1_{\{\rho_t(y^t)\neq i\}}|y^{t-1}] \tag{3}$$

at each y^{t-1}.

By backward induction, this procedure gives us a subgame-perfect equilibrium $\bar{s}$ of $G(C,\rho)$. According to $\bar{s}$, at any y^t, an agent $j \neq \rho_t\left(y^t\right)$ accepts an offer $x_t[y^t]$ iff $E^j[u^j(x^j_t[y^t])|y^t] \geq V^j_{t+1}(y^t)$, and agent $i = \rho_t\left(y^t\right)$ offers $\hat{x}^i_t[V_{t+1},i;\cdot]$, where the process $V = \{V_t\}_{t\in T}$ is the unique solution of (2-3). Our next theorem states that $\bar{s}$ is essentially the only equilibrium.

Theorem 1 *Let C be Pareto-complete. Then, the strategy profile $\bar{s}$ is a subgame-perfect equilibrium of game $G(C,\rho)$. Moreover, for any subgame-perfect equilibrium of $G(C,\rho)$, the vector of continuation values at the beginning of any date t with y^{t-1} (before Y_t is observed) is $V_t\left(y^{t-1}\right)$. After Y_t is observed, the continuation value of an agent i is $m^i\left(V_{t+1}(y^t);y^t\right)$ if $\rho_t\left(y^t\right) = i$, and $V^i_{t+1}(y^t)$ otherwise.*

That is, any equilibrium is payoff equivalent to $\bar{s}$. Moreover, according to $\bar{s}$, the agents reach an agreement $t = 0$, so long as the space of feasible contracts is Pareto-complete, e.g., so long as all side-bets and all risk-sharing agreements are feasible. (Therefore, the delay results cited in the Introduction are based on the authors' exclusion of some of these contracts.)

In any equilibrium, by Theorem 1, when agents learn which of them is to make the first offer, the continuation values are $m^i\left(V_1(y^0); y^0\right)$ and $V_1^j(y^0)$ where $\rho_0\left(y^0\right) = i \neq j$. By Lemma 2.1, this is a Pareto-optimal expected-utility level. Since the utility functions are strictly quasi-concave, it is uniquely obtained by $\hat{x}_0[V_1, \rho_0\left(y^0\right); y^0]$. Therefore, $\left\{\hat{x}_0[V_1, \rho_0\left(y^0\right); y^0]\right\}_{y^0}$ is the unique equilibrium allocation. It is Pareto optimal at y^0, when they learn which player makes the first offer. Before they learn y^0, however, agents are uncertain of which $\hat{x}_0[V_1, \rho_0\left(y^0\right); y^0]$ will prevail. When $\hat{x}_0[V_1, \rho_0\left(y^0\right); y^0]$ varies as Y_0 takes different values of y_0, the uncertainty about $\hat{x}_0[V_1, \rho_0(\cdot); \cdot]$ may hurt our risk-averse agents, causing some inefficiency. In order to avoid this ambiguity in measuring efficiency, we will state our efficiency result for the case that Y_0 is constant.

Theorem 2 *Let C be Pareto-complete. Then, $G(C, \rho)$ admits a unique equilibrium allocation, which is Pareto-optimal at y^0. This allocation is Pareto-optimal at the beginning whenever Y_0 is constant.*

We have seen in Example 1 that, when the space of contracts is Pareto-incomplete, some delay may be necessary for reaching an agreement. When the space of the contracts is Pareto complete, there exists a (preemptive) contract at $t = 0$ that imitates such delay, as the delayed equilibrium allocation is unique. Therefore, even though the unique equilibrium allocation is reachable via contracts at $t = 0$, it is not clear that these contract are not preemptive, imitating a delay. Theorem 2 implies that, typically, the contracts written in equilibrium are not preemptive. For, by the syndicate theory, Pareto-optimal contracts use the minimal information about the total wealth and the discrepancy between the agent's beliefs. When there is a common prior, they depend only on the sum $D^1 + D^2$. Our next example clarifies this matter further and motivates our next theorem.

Example 4 *For any large, even integer $\bar{t}$, let $Y = (Y_0 \ldots, Y_{\bar{t}})$ be identically and independently distributed, each Y_t taking values of 0 and 1 with equal probabilities. As in Example 2, take $D = 4\left(Y_{\bar{t}}, 1 - Y_{\bar{t}}\right)$ and set $u^i\left(x^i\right) = \sqrt{x^i}$ for each i. First consider the bargaining procedure with alternating offers where Agent 1 makes offers at even dates. At $\bar{t} - 1$, Agent 2 offers the riskless allocation $(1, 3)$, which is accepted. This allocation is already Pareto-optimal. Hence, there are multiple equilibria yielding the same allocation: at each $t < \bar{t} - 1$, the recognized agent offers the riskless allocation $(1, 3)$, and the other agent accepts this offer only at some $t^* \leq \bar{t} - 1$.*

Now consider the recognition process according to which Agent 1 makes an offer at t iff $y_t = 0$. Note that $p_t^1 = p_t^2 = 1/2$ for each t. Under this procedure there is a unique equilibrium. At $\bar{t} - 1$, if recognized, agents 1 and 2 will offer $(3, 1)$ and $(1, 3)$, respectively. The offer will be accepted. Hence, at the beginning of $\bar{t} - 1$, the risky allocation that gives $(3, 1)$ and $(1, 3)$ with equal probabilities will prevail. Thus, at $\bar{t} - 2$, the recognized agent will offer the other agent his continuation value, $\left(1 + \sqrt{3}\right)/2$ and keep the rest, which is larger than $\left(1 + \sqrt{3}\right)/2$. Therefore, a risky allocation prevails at $\bar{t} - 2$, too, yielding $\left((1 + \sqrt{3})/2, 4 - (1 + \sqrt{3})/2\right)$ and $\left(4 - (1 + \sqrt{3})/2, (1 + \sqrt{3})/2\right)$ with equal probabilities. Repeating this procedure, one can see that, at each date, a risky allocation prevails, leading agents to reach an agreement at the previous date, in which the randomly determined proposer gets a somewhat larger share. Therefore, there is a unique equilibrium, which yields immediate agreement.[6]

Therefore, under this random recognition procedure, the immediate agreement is not merely an imitation of a delay, as delay is no longer a possible equilibrium outcome. This is generally true in our model:

Theorem 3 *Let p be a bargaining procedure with $p_t^i(\rho_s) \in (0, 1)$ throughout. If D is independent of Y_0 and not Pareto-optimal initially, then the unique equilibrium path-of-play is that at $t = 0$ the recognized agent i offers the contract $\hat{x}_0[V_0, i; \phi]$ and the other agent accepts the offer.*

The rationale behind Theorem 3 is as follows. When there are gains from trade, the recognized agent will get some positive rent at some history y^t. Since either agent is likely to be recognized at t, at the previous date there is a need for insurance against the risk associated with the allocation of this rent. When only one of the agents can get a positive rent when he is recognized, he faces a risk of whether he will be recognized so that he will get the rent, a risk that can be shared. When both agents can get a positive rent when they are recognized, they face uncertainty about which agent will get the rent; and once again they would like to share the risk. In either case, there is a gain to be realized at $t-1$, allocating a positive rent to the agent recognized at $t - 1$. Applying the same argument inductively, one can conclude that there are gains to be realized at 0, in which case delaying agreement will cause an inefficiency (when the first offer made), which cannot be an equilibrium outcome according to Theorem 2. That is, when there are gains from trade, a strictly stochastic bargaining procedure will generate a positive risk (and thus a positive cost of delaying agreement) throughout the game, rendering delay impossible as an equilibrium outcome.

[6]Note that, as $\bar{t} \to \infty$, the equilibrium allocation approaches to $(2, 2)$, while the alternating-offer procedure always yields $(1, 3)$. These two procedures thus yield very different outcomes in our model – whereas they yield similar outcomes in Rubinstein's (1982) model with discounting.

Summary In our model the set of feasible outcomes does not necessarily shrink as time passes. Yet, we are able to pin down a unique (Pareto-optimal) allocation as an equilibrium outcome. Here, revelation of some information makes certain contracts *individually irrational* for some parties, and therefore not viable to society as a whole, a phenomenon known as the Hirshleifer effect. This happens to such an extent that at date $\bar{t}$ the only individually rational allocation is D itself. This sets further restrictions on the set of viable allocations at the previous date, through Lemmas 2 and 3. The inductive application of this procedure gives us a unique (Pareto-optimal) level of utilities at the beginning of the game. Risk-aversion shows up at this stage and tells us that the contract must also be unique, and further requires that, generically, there is no delay in equilibrium.

Equilibrium payoffs and rents In any equilibrium, given any y^t, an agent $i \in N$ gets $m^i\left(V_{t+1}(y^t); y^t\right)$ if he is to make an offer at y^t, and gets only $V^i_{t+1}(y^t)$ if he is not to make an offer at y^t. Hence, the recognized agent $i = \rho_t\left(y^t\right)$ extracts a (non-informational) rent

$$R^i_t\left(y^t\right) = m^i\left(V_{t+1}(y^t); y^t\right) - V^i_{t+1}(y^t) \geq 0. \tag{4}$$

Together with D, these rents determine the equilibrium payoffs. Substituting (4) in (3), we obtain $V^i_t\left(y^{t-1}\right) = E^i\left[R^i_t\left(y^t\right) 1_{\{\rho_t(y^t)=i\}}|y^{t-1}\right] + E^i\left[V^i_{t+1}\left(y^t\right)|y^{t-1}\right]$. Since $V_{\bar{t}} = \left(u^1\left(D^1\right), u^2\left(D^2\right)\right)$, this yields

$$V^i_t\left(y^{t-1}\right) = \sum_{s=t}^{\bar{t}-1} E^i\left[R^i_s\left(y^s\right) 1_{\{\rho_s(y^s)=i\}}|y^{t-1}\right] + E^i\left[u^i\left(D^i\right)|y^{t-1}\right] \tag{5}$$

for each $t \in T$. Equation (5) displays the determinants of bargaining power in this environment. It states that the continuation value of an agent is the sum of the rents he expects to extract in the future when he is recognized, plus his expected utility from consuming the dividend his own asset pays. These rents will be decoupled in our next section.

5 Equilibrium in a Canonical Case

In order to illustrate some of the basic ideas in this paper, we analyze the equilibrium in the following canonical case. We assume that our agents have a common prior and the credit constraints are not binding. i.e., the set of feasible consumption levels is $\mathbb{R}$.

We take $u^i(x) = -\exp(-\alpha_i x)$ for some $\alpha_i > 0$ for each $i \in N$. We let $Y = \{Y_t : t \in T\}$ be independently distributed where each Y_t takes values in $\mathbb{R}^2$, $E[Y_t] = 0$ at every $t > 0$, and $Y_0 \in \mathbb{R}^2$ is a constant. We take $D = Y_0 + Y_1 + \cdots + Y_{\bar{t}}$ so that as time passes agents learn the independent increments in the dividends. Finally, we take a purely deterministic bargaining procedure, and write $T_i(t) = \{s \in T \mid \rho_s = i,\ s \geq t\}$ for the set of all future dates at which agent i is to make an offer.

Notation The total wealth is denoted by $W = D^1 + D^2$. We define the certainty equivalent $CE_t^i[x^i]$ of any x^i for any agent i at any date t as $CE_t^i[x^i] = \left(u^i\right)^{-1}\left(E^i\left[u^i\left(x^i\right)|Y_0,\ldots,Y_t\right]\right)$. When x^i is a function of $(Y_{t+1},\ldots,Y_{\bar{t}})$, it is stochastically independent of $(Y_0,\ldots,Y_t)$, and hence

$$CE_t^i[x^i] = -\frac{1}{\alpha_i}\log E[\exp(-\alpha_i x^i)] \equiv M[x^i](\alpha_i). \tag{6}$$

In Figure 1, we plot the equilibrium contracts $\{\hat{x}_t\}_{t\in T}$ in terms of their certainty equivalents computed at date 0. In this figure, the vector of certainty equivalents of any given allocation x_t of the form $x_t = E[D|Y_0,\ldots,Y_t] + f(Y_{t+1},\ldots,Y_{t-1})$ stays under the line l_t, where $E[D|Y_0,\ldots,Y_t] = Y_0 + Y_1 + \cdots + Y_t$. Note that l_t moves inwards as t increases. At $\bar{t}$, they consume D, which is depicted at the origin. Assume that Agent 2 makes an offer at $\bar{t}.- 1$. He offers to share the unresolved risk $Y_{\bar{t}}^1 + Y_{\bar{t}}^2$ optimally, and takes all the rent. The certainty equivalent of this allocation is found by going up until we reach the line $l_{\bar{t}-1}$. At $\bar{t}-2$, Agent 1 now offers to share the risk $Y_{\bar{t}-1}^1 + Y_{\bar{t}-1}^2$ as well. He now extracts all the rent of agreeing one day earlier. Hence, the vector of certainty equivalents for this contract is found by going horizontally until we reach the line $l_{\bar{t}-1}$. Using this algorithm until we reach the line l_0, we find the certainty equivalents of the contract signed at $t = 0$.

We compute that the equilibrium allocation at any date $t \in T$ is

$$\begin{aligned}\hat{x}_t^i = E[D^i|Y_0,\ldots,Y_t] \;&+\; \frac{\alpha_j}{\alpha_1+\alpha_2}(W - E[W|Y_0,\ldots,Y_t]) \\ &- \sum_{s\in T_j(t)} M\left[\frac{\alpha_j}{\alpha_1+\alpha_2}(Y_{s+1}^1 + Y_{s+1}^2)\right](\alpha_i) - M\left[Y_{s+1}^i\right](\alpha_i) \\ &+ \sum_{s\in T_i(t)} M\left[\frac{\alpha_i}{\alpha_1+\alpha_2}(Y_{s+1}^1 + Y_{s+1}^2)\right](\alpha_j) - M\left[Y_{s+1}^j\right](\alpha_j),\end{aligned} \tag{7}$$

where $i,j \in N$ with $i \neq j$. (One can easily check the validity of (7) by induction.) Note that the share of the social risk $W - E[W|Y_0,\ldots,Y_t]$ borne by agent i at any optimal risk-sharing scheme is $\frac{\alpha_j}{\alpha_1+\alpha_2}(W - E[W|Y_0,\ldots,Y_t])$. At any $s \in T_i(t)$, in order to insure himself against his individual risk Y_{s+1}^j, agent j is offered to bear $\frac{\alpha_i}{\alpha_1+\alpha_2}(Y_{s+1}^1 + Y_{s+1}^2)$ in addition to the risk $\frac{\alpha_i}{\alpha_1+\alpha_2}(W - E[W|Y_0,\ldots,Y_{s+1}])$ that he would bear if they were to agree at the next date. Since the agent i makes an offer that can be rejected only by delaying the agreement until $s+1$, he extracts the entire value $M\left[\frac{\alpha_i}{\alpha_1+\alpha_2}(Y_{s+1}^1 + Y_{s+1}^2)\right](\alpha_j) - M\left[Y_{s+1}^j\right](\alpha_j)$ of this transaction for agent j. This value is the certainty equivalent of the extra risk that agent j bears, minus the certainty equivalent of the risk that agent j avoids. Thus, (7) is read as: The consumption allocated to an agent at some t is the certainty equivalent of his own asset's dividend *plus* the optimal risk that he shares, *minus* the rents that he pays to the other agent when the other agent makes an offer (in the future), *plus* the rent that he extracts from the other agent when he makes an offer (in the future).

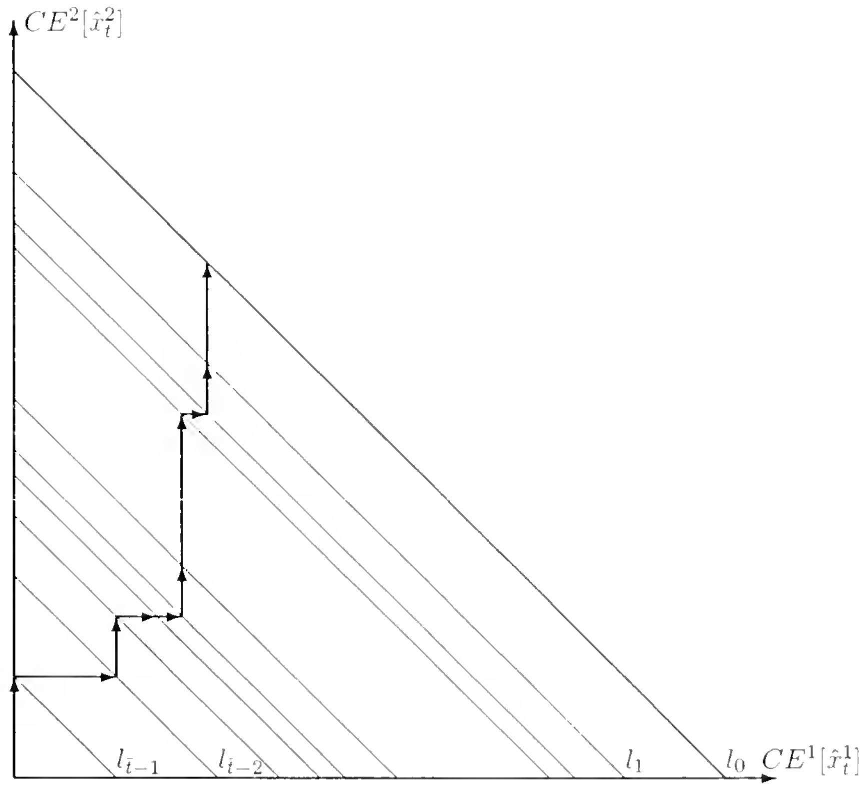

Figure 1: The sequence of the equilibrium contracts in the space of certainty equivalents computed at $t = 0$.

Using (7), we compute[7] that

$$CE_t^i[\hat{x}_t^i] = CE_t^i[D^i] + \sum_{s \in T_i(t)} G_s, \tag{8}$$

where $CE_t^i[D^i] = E[D^i|Y_0, \ldots, Y_t] + \sum_{s \geq t} M\left[Y_{s+1}^i\right](\alpha_i)$ and

$$G_s = M\left[Y_{s+1}^1 + Y_{s+1}^2\right]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) - \left[M\left[Y_{s+1}^j\right](\alpha_j) + M\left[Y_{s+1}^i\right](\alpha_i)\right]. \tag{9}$$

Given that they will share the risk optimally, the value of the increment $Y_{s+1}^1 + Y_{s+1}^2$ in total wealth for the agents at date s is $M\left[Y_{s+1}^1 + Y_{s+1}^2\right]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)$, the certainty equivalent of $Y_{s+1}^1 + Y_{s+1}^2$ with respect to *the group's surrogate utility-function* of Wilson (1968). If they were not able to share the risk, the value of the increments Y_{s+1}^i and Y_{s+1}^j in the social wealth would be only $M\left[Y_{s+1}^j\right](\alpha_j) + M\left[Y_{s+1}^i\right](\alpha_i)$. Therefore, G_s is the gains from sharing the risk in Y_{s+1}, optimally. Therefore, (8) states that, in CE terms, the consumption that our equilibrium allocates to agent i at date t consists of what he already owns and the sum of gains from (optimal) trade when he makes offers in the future. Equation (8) is similar to (5). Moreover, in CE space we have transferable utility, which allows us to decouple the gains from trade at each date from the rest of the problem.

Observations Firstly, *being able to make an offer prior to an important information-revelation strengthens the position of a bargainer.* To see this, let $Y = (Y_0, \ldots, Y_s, Y_{s+1} + Z, Y_{s+2}, \ldots, Y_{\bar{t}})$ and $Y^* = (Y_0, \ldots Y_{s^*}, Y_{s^*+1} + Z, Y_{s^*+2}, \ldots, Y_s, Y_{s+1}, Y_{s+2}, \ldots, Y_{\bar{t}})$ be such that $\{Y_1, \ldots, Y_{\bar{t}}, Z\}$ is independently distributed and $\rho_{s^*} \neq \rho_s$. If we replace Y with Y^*, then $CE^{\rho_{s^*}}[\hat{x}_0^{\rho_{s^*}}]$ increases by $G[Z] = M\left[Z^1 + Z^2\right]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right) - \left[M\left[Z^j\right](\alpha_j) + M\left[Z^i\right](\alpha_i)\right]$, while $CE^{\rho_s}[\hat{x}_0^{\rho_s}]$ decreases by the same amount.

Second, if we replace a bargaining procedure p with $\hat{p}$ by changing $p_{t^*}^i = 0$ to $\hat{p}_{t^*}^i = 1$ at some t^* for some i, then $CE^i[\hat{x}_0^i]$ increases by G_{t^*}. [Cf. Theorem 4 below.]

Third, *any Pareto-optimal allocation is approximately obtainable via a deterministic bargaining procedure as long as information arrives smoothly.* Assume that $Y_1, \ldots, Y_{\bar{t}}$ are identically distributed with variance $\sigma^2/\bar{t}$. Then, given any Pareto-optimal and individually rational allocation $x \in X$ and any $\epsilon > 0$, there exists a $\hat{t}$ such that we can find a deterministic bargaining procedure p that yields $\left|CE^i[\hat{x}_0^i] - CE^i[x^i]\right| < \epsilon$ for each $i \in N$ as long as $\bar{t} > \hat{t}$. To do this, letting $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1\left|CE^1[x^1] - CE^1[D^1]\right| = \lambda_2\left|CE^2[x^2] - CE^2[D^2]\right|$ and $\lambda_1 + \lambda_2 > 0$, we write n for the integer part of $\frac{\lambda_2}{\lambda_1+\lambda_2}\bar{t}$. We consider the bargaining procedure p with $p_t^1 = 1$ iff $t \leq n$. Now, by (8), $CE^1[\hat{x}_0^1] - CE^1[D^1] = nG_0 \equiv (n/\bar{t})\,G$

[7] By independence, $CE_t^i[\frac{\alpha_j}{\alpha_1+\alpha_2}(W - E[W|Y_0, \ldots, Y_t])] = \sum_{s \geq t} M\left[\frac{\alpha_j}{\alpha_1+\alpha_2}(Y_{s+1}^1 + Y_{s+1}^2)\right](\alpha_i)$. Also, $M\left[\frac{\alpha_2}{\alpha_1+\alpha_2}x\right](\alpha_1) + M\left[\frac{\alpha_1}{\alpha_1+\alpha_2}x\right](\alpha_2) = M[x]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)$ at any x. Substitute these equalities in (7).

and $CE^2[\hat{x}_0^2] - CE^2[D^2] = (1 - n/\bar{t})\,G$, where $G = \sum_s G_s$ is the total gain from trade. Thus, $\left|CE^i[\hat{x}_0^i] - CE^i[x^i]\right| = \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\bar{t} - n\right) G/\bar{t} \leq G/\bar{t}$, where the right hand side goes to 0 as $\bar{t} \to \infty$. [Cf. Theorem 5 below.]

The impact of risk aversion Many axiomatic bargaining solutions have the property that, *if a bargainer gets more risk-averse, then he becomes worse off, while his opponent benefits from this.* Examples of such bargaining solutions are Nash (1950), Mashler-Perles (1980) and Kalai-Smorodinsky (1975). Roth (1985) shows that Rubinstein's bargaining model has the same property if we consider "instantaneous risk-aversion."[8] We will now show that this property does not hold in our model, where the bargaining is driven by risk aversion itself.[9]

In our model, *risk-aversion of an agent may hurt or benefit his opponent.* To see this, note that an increase in α_j affects $CE_t^i[\hat{x}_t^i]$ through G_s for $s \in T_i(t)$. As α_j increases, on the one hand, $M\left[Y_{s+1}^j\right](\alpha_j)$ decreases, which increases G_s. On the other hand, $M\left[Y_{s+1}^1 + Y_{s+1}^2\right]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)$ decreases, which results in a decrease in G_s. Either of these effects can be dominant: If $D^1 + D^2$ is constant, the latter is identically 0, hence, the risk aversion of j benefits i. If D^j is a constant, the former is identically 0, thus an increase in α_j renders $CE_t^i[\hat{x}_t^i]$ lower, i.e., risk-aversion of j hurts i. Intuitively, the decrease in $M\left[Y_{s+1}^1 + Y_{s+1}^2\right]\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)$ is the loss that agent j's risk-aversion causes to society, part of which is borne by agent i; and the decrease in $M\left[Y_{s+1}^j\right](\alpha_j)$ is the decrease in j's reservation value, which strengthens i's position in bargaining.

In order to demonstrate which effect is dominant in which cases, let Y_{s+1} be a bivariate normal random vector with variance-covariance matrix $\begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$. Then, we have $M\left[Y_{s+1}^i\right](\alpha) = -\frac{1}{2}\alpha\sigma_i^2$ and $M\left[Y_{s+1}^1 + Y_{s+1}^2\right](\alpha) = -\frac{1}{2}\alpha(\sigma_1^2 + \sigma_2^2 + 2r\sigma_1\sigma_2)$. Substituting these equalities into (9) and differentiating with respect to α_j, we find that G_s (and hence $CE_t^i[\hat{x}_t^i]$) is increasing in α_j iff

$$r < \bar{r} \equiv \left[\frac{1}{2}\left(\frac{\alpha_j}{\alpha_i}\right)^2 + \frac{\alpha_j}{\alpha_i}\right]\frac{\sigma_j}{\sigma_i} - \frac{1}{2}\frac{\sigma_i}{\sigma_j}. \tag{10}$$

[8] See Roth (1985) for the references to these results. In contrast with these results, White (1999) shows that adding a non-contractible background-noise to an agent's consumption would make him better off, leaving his opponent worse off.

[9] One can check that the certainty equivalent of an agent is decreasing with his own risk-aversion. But this hardly implies that he is in a weaker bargaining position, as the certainty equivalent of his consumption would still decrease even if his consumption had not changed at all. In comparing a player's equilibrium welfare-levels as his own risk-aversion changes, we clearly perform an interpersonal comparison between the two selves of the agent, each self having a different utility function. We can safely say that an agent gets worse off as he becomes more risk-averse, only if each self finds the agent's consumption worse under higher risk-aversion. Since the equilibrium outcome is Pareto optimal, the more risk-averse self will find himself worse off only when his risk-aversion benefits his opponent.

Here, an agent benefits from the risk aversion of his opponent when r is small enough, the case when the social cost of this risk-aversion is not significant. Moreover, the upper bound $\bar{r}$ increases with both α_j/α_i and σ_j/σ_i. An intuition for this observation comes from insurance: As an insured becomes more risk-averse, we expect the insurer to benefit. On the other hand, if the insurer is the one who becomes more risk-averse, we would not expect the other party –the insured– to benefit. As α_j/α_i and σ_j/σ_i increase, agent i becomes the insurer and benefits from the risk aversion of agent j.

6 Monotonicity and Obtainability

In this section, we present two results about the equilibrium. The first one is an extension of a result in Merlo and Wilson (1995) and Yildiz (2001) to our model. It states that an agent cannot lose when he becomes more likely to make an offer in the future. This fact will also help us in proving our second result. The second result states that any individually rational and Pareto optimal allocation is obtainable as an equilibrium outcome of some bargaining procedure iff "the first-mover advantage" is not too large, i.e., the uncertainty resolved at $t = 1$ is not substantial.

Given any $p \in \mathcal{P}$, let us write $U_0(p)$ for the vector V_0 of equilibrium-continuation values at the beginning for game $G(C, p)$ where C is Pareto-complete.

Theorem 4 *Given any two bargaining procedures $p,\hat{p} \in \mathcal{P}$ with $p_t^i \geq \hat{p}_t^i$ at each $t \in T$ for some $i \in N$, we have $U_0^i(p) \geq U_0^i(\hat{p})$ and $U_0^j(p) \leq U_0^j(\hat{p})$ for $j \neq i$.*

Merlo and Wilson (1995) provide a counter-example, showing that this result cannot be extended to more than two agents in their model. An increase in the probability of recognition for an agent, say 1, decreases the continuation value of another agent, say 3, which may in turn increase the continuation value of some other agent, say 2; and this may hurt agent 1. Using an example in which the agents may have different beliefs about the recognition process, Yildiz (2001) shows that the conclusion of this teorem would not be valid if the recognition process were not affiliated.

Given any (Pareto-optimal and individually rational[10]) allocation $x \in X$, can we find a bargaining procedure $p{\in}\mathcal{P}$ that gives us x as its equilibrium allocation (of game $G(C,p)$)? If such a bargaining procedure p exists, then we will say that x is *obtainable* (*via* p).

We will now answer this question. To this end, take any $i \in N$, and set $\mathcal{P}_i = \{p{\in}\mathcal{P} | p_0^i = 1\}$. The best bargaining procedure for agent i (by Theorem 4) is $\overline{p}[i] \in \mathcal{P}_1$ that recognizes agent i with probability 1 at each history and date. Under $\overline{p}[i]$, agent i extracts all the rent, leaving agent j with his reservation utility-level. On the other hand, by Theorem 4, the bargaining

[10] That is, $E^i\left[u^i\left(x^i\right)\right] \geq E^i\left[u^i\left(D^i\right)\right]$ for each i.

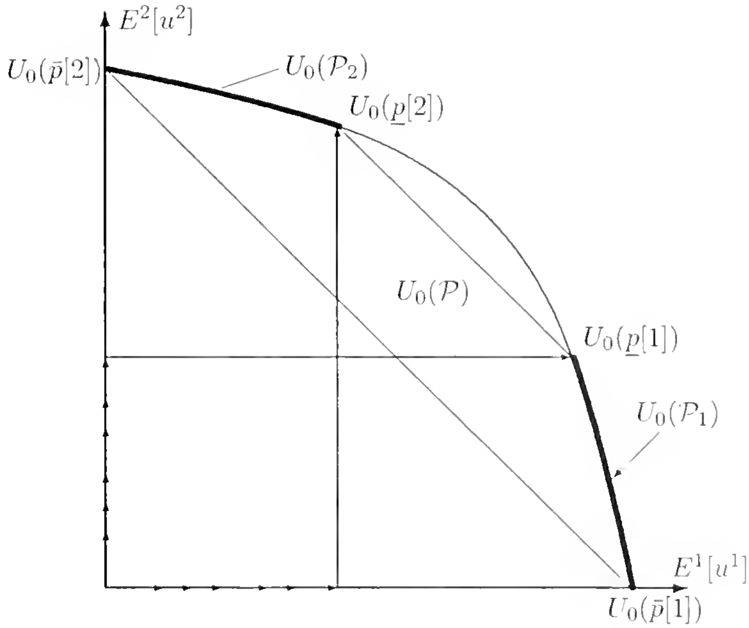

Figure 2: Obtainable utility levels. The disagreement utilities are at the origin.

procedure minimizing U_0^i over $\mathcal{P}_i$ is $\underline{p}[i] \in \mathcal{P}_i$ where $\underline{p}[i]_t^i$ is 1 when $t = 0$, and 0 otherwise. Under $\underline{p}[i]$, i makes only the first offer. (In that case, at date $t = 1$, agent j can extract the entire rent, leaving agent i indifferent to consuming D, when both agents know y^1. At date 0, agent i extracts the remaining rent, which is associated with the uncertainty in y^1.)

Theorem 5 *Let C be Pareto-complete. Every Pareto-optimal and individually-rational allocation is obtainable iff $U_0^i\left(\underline{p}\,[i]\right) \leq U_0^i\left(\underline{p}\,[j]\right)$ for some $i \neq j$.*

The condition that $U_0^i\left(\underline{p}\,[i]\right) \leq U_0^i\left(\underline{p}\,[j]\right)$ for some $i \neq j$ expresses that it is not the case that, for some agent, the worst outcome when he moves first is better than the best outcome when he does not, i.e., *the first-mover advantage is not too large.* Therefore, Theorem 5 can be read as: Every Pareto-optimal and individually-rational allocation is obtainable so long as the first-mover advantage is not too large. The first-mover advantage is large when a substantial part of uncertainty resolved at date $t = 1$. If this is the case, then many axiomatic bargaining solutions that select centrally located utility pairs, such as the Nash bargaining solution, will not be obtainable.

When uncertainty is resolved smoothly, the first-mover advantage will be small, and therefore every Pareto-optimal and individually-rational allocation will be obtainable.

The proof is illustrated in Figure 2. Given any i, since U_0 is continuous and $\mathcal{P}_i$ is connected, $U_0(\mathcal{P}_i)$ is a connected set with the end-points $U_0^i(\underline{p}[i])$ and $U_0^i(\bar{p}[i])$, which we computed above. By Theorem 2, $U_0(\mathcal{P}_i)$ is on the Pareto-frontier, hence it is the part of Pareto-frontier connecting these two end-points. Moreover, any bargaining procedure $p \in \mathcal{P}$ can be written as a convex combination of two such bargaining procedures $p[1] \in \mathcal{P}_1$ and $p[2] \in \mathcal{P}_2$, where $p[j]$ is the same as p after j is recognized at 0, but recognizes j at 0 with certainty. Then, by linearity, $U_0(\mathcal{P})$ is the convex hull of $U_0(\mathcal{P}_1) \cup U_0(\mathcal{P}_2)$, as shown in the figure. Since the agents are strictly risk averse, the Pareto-frontier is strictly concave, hence the set of Pareto-optimal payoff vectors in $U_0(\mathcal{P})$ is $U_0(\mathcal{P}_1) \cup U_0(\mathcal{P}_2)$. This set contains all the Pareto-optimal and individually rational payoff vectors if and only if $U_0(\underline{p}[2])$ is to the right of $U_0(\underline{p}[1])$, the characterizing condition in our Theorem.

7 Conclusion

Consider a two-person risk-sharing problem where the agents' beliefs may differ but there is no private information. Assuming that a Pareto-optimal sharing rule is employed, Wilson (1968), in his seminal paper "The theory of syndicates", shows that the group can be represented by an expected-utility maximizer only if the agents hold the same beliefs or the sharing rule happens to be linear. Since there is a continuum of Pareto-optimal and individually rational allocations, we would expect the agents to allocate the risk through negotiation. But, if the agents hold different beliefs, the negotiation may yield Pareto-suboptimal outcomes. Therefore, the Pareto-optimality assumption of Wilson (1968) may not hold.

In this paper, confining ourself to sequential bargaining procedures with immediately expiring offers, we show that the equilibrium allocation will be Pareto-optimal and that there will be no delay for generic cases, provided there is a sufficiently rich set of contracts to generate the Pareto-optimal allocations throughout the negotiation –an implicit assumption in Wilson (1968). This result strengthens Wilson's theory by predicting its Pareto-optimality assumption.

The allocation chosen by the group in equilibrium depends on the way uncertainty is to be resolved. For example, in Section 5, where the agents hold the same beliefs and the sharing rule is linear, revealing an informative signal at some time t rather than s is equivalent to transferring the gain from trade associated with this signal from the agent who makes an offer at $s-1$ to the agent who makes an offer at $t-1$. Since the decision of an expected utility maximizer is typically independent of such details, this suggests that we may not be able to find an expected-utility maximizer that represents the group even if the agents hold the same beliefs. This may strengthen Wilson's negative result.

A Proofs

This section contains the proofs omitted in the text. Throughout these proofs, we will use the fact that C is Pareto-complete iff, given any $x \in X$ and any y^t, there exists a feasible allocation $\tilde{x} \in C_t$ such that $E^i[u^i(\tilde{x}^i)|y^t] \geq E^i[u^i(x^i)|y^t]$ at each $i \in N$.

Towards proving Lemma 1, write each P^i as $\tilde{P} \times \bar{P}^i$, where $\tilde{P}$ is the common prior associated with $\tilde{Y}$. Given any (i, x^i, y^t), we define the conditional marginals $\bar{E}^i\left[x^i|y^t\right](\tilde{y}) = \int_{\bar{y}} x^i(\tilde{y}, \bar{y})\, d\bar{P}^i(\cdot|y^t)$ and $\tilde{E}^i\left[x^i|y^t\right](\bar{y}) = \int_{\tilde{y}} \int_{\bar{y}} x^i(\tilde{y}, \bar{y})\, d\tilde{P}(\cdot|y^t)$.

Proof. (*Lemma 1*) Take any y^t and any $x \in X$. Since $x^1 + x^2 \leq D^1 + D^2$, $\tilde{E}\left[x^1|y^t\right] + \tilde{E}\left[x^2|y^t\right] \leq \tilde{E}\left[D^1|y^t\right] + \tilde{E}\left[D^1|y^t\right]$. Moreover, when $D \in \bar{X}$, $\tilde{E}\left[D^i|y^t\right] \equiv D$ for each $i \in N$, hence $\tilde{E}\left[x^1|y^t\right] + \tilde{E}\left[x^2|y^t\right] \leq D^1 + D^2$. Thus, $\left(\tilde{E}\left[x^1|y^t\right], \tilde{E}\left[x^2|y^t\right]\right) \in \bar{X}$. But, by Jensen's inequality, $E^i\left[u^i(x^i)|y^t\right] = \bar{E}^i[\tilde{E}\left[u^i\left(x^i\right)|y^t\right]|y^t] \leq \bar{E}^i[u^i(\tilde{E}\left[x|y^t\right])|y^t] = E^i[u^i(\tilde{E}\left[x|y^t\right])|y^t]$ for each $i \in N$. Therefore, $\left(\bar{X}, \ldots, \bar{X}\right)$ is Pareto-complete. ∎

Proof. (*Lemma 2*) Since (u^1, u^2) is increasing, the optimization problem (1) corresponds to the optimization problem

$$\max_{w \in \tilde{U}(C_t; y^t)} w^i \quad \text{subject to} \quad w^j \geq v^j \tag{11}$$

in the utility space where $\tilde{U}(C_t; y^t) = \{w \in \mathbb{R}^2 | \exists x[y^t] \in C_t[y^t] \text{ s.t. } w^i \leq E^i[u^i(x^i[y^t])|y^t]\ \forall i \in N\}$. Firstly, $\tilde{U}(C_t; y^t) = \tilde{U}(X; y^t)$ for each C_t,[11] hence, if exists, the solution $m^i(v; y^t)$ to (11) does not depend on C_t. Moreover, $U(X; y^t) = \{(E^1[u^1(x^1)|y^t], E^2[u^2(x^2)|y^t])|x \in X\}$ is compact.[12] Hence, a solution $m^i(v; y^t)$ to (11) (and thus a solution $\hat{x}_t[v, i; y^t]$ to (1)) exist. As we will see, $\hat{x}_t[v, i; y^t]$ is Pareto optimal at y^t, hence $\hat{x}_t[v, i; y^t]$ is unique (almost surely), and is in $C_t[y^t]$.

In the following, without loss of generality, we will take $C_t = X$. Towards proving part 1, suppose that $\hat{x}_t[v, i; y^t]$ is not Pareto optimal at y^t. Then, there exists some $\tilde{x}[y^t] \in X[y^t]$ such that $E^i[u^i\left(\tilde{x}^i[y^t]\right)|y^t] \geq E^i[u^i(\hat{x}_t^i[v, i; y^t])|y^t] = m^i(v; y^t)$ and $E^j[u^j\left(\tilde{x}^j[y^t]\right)|y^t] \geq E^j[u^j(\hat{x}_t^j[v, i; y^t])|y^t]$, and one of the inequalities is strict. Since $\hat{x}_t[v, i; y^t]$ is a solution to the problem (1), we also have $E^j[u^i(\hat{x}_t^j[v, i; y^t])|y^t] \geq v^j$, and hence we have $E^j[u^j\left(\tilde{x}^j[y^t]\right)|y^t] \geq v^j$, i.e., $\tilde{x}[y^t]$ is feasible to the optimization problem. Thus, $E^i[u^i\left(\tilde{x}^i[y^t]\right)|y^t] \leq m^i(v; y^t)$. Therefore, $E^i[u^i\left(\tilde{x}^i[y^t]\right)|y^t] = m^i(v; y^t)$. Since at least one of the inequalities was strict, this implies that $E^j[u^j\left(\tilde{x}^j[y^t]\right)|y^t] > v^j$. Since u^j is increasing and continuous, it follows that there exists some random variable $\delta > 0$ such that $\tilde{x}^j[y^t] - \delta \geq \underline{x}^i$ and $E^j[u^j\left(\tilde{x}^j[y^t] - \delta\right)|y^t] \geq v^j$. But then, for $(\tilde{x}^i[y^t] + \delta, \tilde{x}^j[y^t] - \delta) \in X[y^t]$, we have $E^i[u^i\left(\tilde{x}^i[y^t] + \delta\right)|y^t] > m^i(v; y^t)$, which is a contradiction. Therefore, $\hat{x}_t[v, i; y^t]$ is Pareto-optimal at y^t. Part 2 of the lemma can be proven by similar arguments to the last one. As for Part 3, since $z \in X$, $m^i(v; y^t) \geq v^i$. Moreover, since $\hat{x}_t[v, i; y^t]$ is Pareto-optimal at y^t and $E^j[u^j\left(\hat{x}_t^j[v, i; y^t]\right)] = E^j[u^j\left(z^j\right)|y^t]$, if $m^i(v; y^t) = v^i$, then z will be Pareto-optimal at y^t, showing the contrapositive of the last statement. ∎

[11] To see this, take any $w \in \tilde{U}(X; y^t)$ so that $w \leq \left(E^1[u^1(x^1)|y^t], E^2[u^2(x^2)|y^t]\right)$ for some $x \in X$. Since $\mathbf{C}$ is Pareto-complete, there exists some $\tilde{x} \in C_t$ such that $E^i[u^i(x^i)|y^t] \leq E^i[u^i(\tilde{x}^i)|y^t]$ for each $i \in N$. Thus, $w \leq \left(E^1[u^1(\tilde{x}^1)|y^t], E^2[u^2(\tilde{x}^2)|y^t]\right)$, showing that $w \in \tilde{U}(C_t; y^t)$. Therefore, $\tilde{U}(X; y^t) \subseteq \tilde{U}(C_t; y^t)$. On the other hand, $C_t \subseteq X$, thus $\tilde{U}(C_t; y^t) \subseteq \tilde{U}(X; y^t)$. Therefore, $\tilde{U}(C_t; y^t) = \tilde{U}(X; y^t)$.

[12] Since u^1 and u^2 are continuous and $X[y^t]$ is closed, $U(X; y^t)$ is closed. Moreover, given any $x \in X$, $x \geq \underline{x}$. Hence, $x \leq x_{huge} = (D^1 + D^2 - \underline{x}^2, D^1 + D^2 - \underline{x}^1)$. Since u^i is increasing, it follows that $u^i\left(\underline{x}^i\right) \leq E^i[u^i(x^i)|y^t] \leq E^i[u^i(x_{huge}^i)|y^t]$. Therefore, $U(X, y^t)$ is bounded.

Proof. (*Lemma 3*) Let us write $\tilde{x}[y^s] \equiv \hat{x}_s[v, i; y^s]$ for each y^s. We want to select $\tilde{x}[y^s]$ in such a way that $\tilde{x} \in X$. By construction, $\tilde{x} \geq \underline{x}$ and $\tilde{x}^1 + \tilde{x}^2 \leq D^1 + D^2$. Hence, we only need to guarantee that $\tilde{x}^k$ is integrable for each $k \in N$. Now, by Lemma 2.1, $\tilde{x}[y^s]$ is Pareto-optimal. Thus, by the syndicate theory, there exists some $\lambda_0 > 0$ such that the equations

$$\tilde{x}(y;\lambda) = \arg \max_{\substack{x^1 + x^2 \leq W(y) \\ x^k \geq \underline{x}^k}} u^1(x^1) P^1(y|y^s) + \lambda u^2(x^2) P^2(y|y^s) \tag{12}$$

(at each continuation y of y^s) and

$$\Phi(\lambda) \equiv \int u^j(\tilde{x}^j(y;\lambda)) dP^j(y|y^s) = E^j[u^j(z^j)|y^s] \tag{13}$$

hold when $\lambda = \lambda_0$. We set $\tilde{x}(y) = \tilde{x}(y;\lambda_0)$. [We choose $\tilde{x}(y;\lambda)$ to satisfy Equation (12) at each y rather than almost surely. Hence the phrase "we can choose" in the statement of Lemma.] Clearly, $\tilde{x}(y;\lambda)$ is continuous in λ. Hence, so is Φ. Thus, λ_0 that solves equation (13) is upper semi-continuous in $v^j = E^j[u^j(z^j)|y^s]$. Since $\tilde{x}(y) \equiv \tilde{x}(y;\lambda_0)$ is continuous in λ_0, it will also be continuous in $v^j = E^j[u^j(z^j)|y_s]$. Since $E^j[u^j(z^j)|y^s]$ is measurable, we conclude that $\tilde{x}$ is measurable. Moreover, $\tilde{x}^k \leq D^1 + D^2 - \underline{x}^l \in X^k$ for $l \in N \setminus \{k\}$ and hence $E^k[|\tilde{x}^k|] \leq |\underline{x}^k| + E^k[|D^1 + D^2 - \underline{x}^l|] < \infty$; and this completes the proof. ■

Proof. (*Theorem 1*) We have already shown in the text that $\bar{s}$ is an equilibrium. Now, given any equilibrium s^*, we will show (via mathematical induction) that the vector of continuation values at the beginning of any t is V_t. This is trivially true at $\bar{t}$. Assume that it is true at some $t+1$. Since s^* is an equilibrium, an agent j must accept an offer $x_t[y^t]$ if $E^j[u^j(x_t^j[y^t])|y^t] > V_{t+1}^j(y^t)$, and reject it if $E^j[u^j(x_t^j[y^t])|y^t] < V_{t+1}^j(y^t)$. Thus, j's action in s^* can differ from his action in $\bar{s}$ only by rejecting an offer $x_t[y^t]$ when $E^j[u^j(x_t^j[y^t])|y^t] = V_{t+1}^j(y^t)$. The only such distinction that can make a difference is when j rejects $\hat{x}_t[V_{t+1}(y^t), i; y^t]$. Let us consider this case. If V_{t+1} is not Pareto-optimal at y^t, then by Lemma 2.3, $m^i(V_{t+1}(y^t); y^t) > V_{t+1}^i(y^t)$ where $i = \rho_t(y^t)$. Since $\hat{x}_t[V_{t+1}(y^t), i; y^t]$ is rejected, the best response correspondence of i at y^t is empty. This contradicts the fact that s^* is an equilibrium. Therefore, when V_{t+1} is not Pareto-optimal at y^t, $\hat{x}_t[V_{t+1}(y^t), i; y^t]$ must be accepted, and hence s^* and $\bar{s}$ yield the same outcome at y^t. On the other hand, when V_{t+1} is already Pareto-optimal at y^t, $m^i(V_{t+1}(y^t); y^t) > V_{t+1}^i(y^t)$. Hence, if j is to reject $\hat{x}_t[V_{t+1}(y^t), i; y^t]$, the best response of i is to make an offer that is to be rejected, yielding $m^i(V_{t+1}(y^t); y^t)$ for i and $V_{t+1}^j(y^t)$ for j – as in $\bar{s}$. Therefore, in either case, s^* and $\bar{s}$ yield the same continuation values at y^t (after the agent is recognized). Therefore, at the beginning of t, they yield the same vector of continuation values, which is $V_t(y^{t-1})$. ■

We now prove Theorem 3. Using the notation of Lemma 1, we take $\tilde{Y} = \rho$, and write each y^t as $(\rho_t, \bar{y}^t)$ and each y as $(\rho, \bar{y})$, where ρ_t is the history of the recognized agents. ($\bar{Y}$ consists of all the information except for the recognition process.) The recognition process and the dividends are stochastically independent, hence $D \in \overline{X}$, i.e., $D(\rho, \bar{y}) = D(\rho', \bar{y})$ for each $(\rho, \rho', \bar{y})$. The allocations generated by contracts below will all be in $\bar{X}$, hence the irrelevant argument ρ will be omitted. We first prove the following Lemma.

Lemma 4 *Let $p \in \mathcal{P}$ be such that $p_t^i(y^s) \in (0,1)$ for each (i, t, y^s). If $R_t^i(y^t) = 0$ for some y^t with $i = \rho_t(y^t)$, then*

$$P^j\left(R_{t+1}^k(y^{t+1}) 1_{\{\rho_{t+1}(y^{t+1}) = k\}} = 0 | y^t\right) = 1$$

for each $j, k \in N$.

Proof. We will prove the contrapositive. That is, assuming that

$$P^j\left(R_{t+1}^k\left(y^{t+1}\right)1_{\{\rho_{t+1}=k\}}\neq 0|y^t\right)>0 \tag{14}$$

for some $j,k\in N$, we will show that $R_t^i\left(y^t\right)>0$, where $i=\rho_t\left(y^t\right)$. Since there are no rents at $\bar{t}$, our assumption holds only for $t\leq\bar{t}-1$. Considering any such t with (14), we take $l\neq k$. We take y^t and its continuations y_k^{t+1} and y_l^{t+1} as generic realizations where $\rho_{t+1}\left(y_k^{t+1}\right)=k$ and $\rho_{t+1}\left(y_l^{t+1}\right)=l$. We will simply write $\hat{x}_{t+1}\left[y_k^{t+1}\right]$ for $\hat{x}_{t+1}\left[V_{t+2}\left(y_k^{t+1}\right),k,y_k^{t+1}\right]$. By Lemma 1, $\hat{x}_{t+1}\left[y_k^{t+1}\right]\in\bar{X}\left[y_k^{t+1}\right]$. We define $\hat{x}_{t+1}\left[y_l^{t+1}\right]$, similarly. Define $\tilde{x}\in\bar{X}$ by setting

$$\tilde{x}(\bar{y})=p_t^k\left(y^t\right)\hat{x}_{t+1}\left[y_k^{t+1}\right](\bar{y})+\left(1-p_t^k\left(y^t\right)\right)\hat{x}_{t+1}\left[y_l^{t+1}\right](\bar{y})$$

at each continuation $(\boldsymbol{\rho},\bar{y})$ of y^t. By concavity, we have

$$\begin{aligned}E^i\left[u^i\left(\tilde{x}^i\right)|y^t\right] &\geq E^i\left[p_t^k\left(y^t\right)u^i\left(\hat{x}_{t+1}^i\left[y_k^{t+1}\right]\right)\left(1-p_t^k\left(y^t\right)\right)+u^i\left(\hat{x}_{t+1}^i\left[y_l^{t+1}\right]\right)|y^t\right]\\ &= p_t^k\left(y^t\right)E^i\left[u^i\left(\hat{x}_{t+1}^i\left[y_k^{t+1}\right]\right)|y^t\right]+\left(1-p_t^k\left(y^t\right)\right)E^i\left[u^i\left(\hat{x}_{t+1}^i\left[y_l^{t+1}\right]\right)|y^t\right]=V_{t+1}^i\left(y^t\right)\end{aligned}$$

for each $i\in N$. We **claim** that $E^j\left[u^j\left(\tilde{x}^j\right)|y^t\right]>V_{t+1}^j\left(y^t\right)$. In that case, $V_{t+1}\left(y^t\right)$ is not a Pareto-optimal utility pair at y^t, thus by Lemma 2.3, $m^i\left(V_{t+1}\left(y^t\right);y^t\right)>V_{t+1}^i\left(y^t\right)$, showing that $R_t^i\left(y^t\right)=m^i\left(V_{t+1}\left(y^t\right);y^t\right)-V_{t+1}^i\left(y^t\right)>0$ for each $i=\rho_t\left(y^t\right)$.

Now we prove our **claim**. Since ρ is affiliated, by Theorem 4, we have $V_{t+2}^k\left(y_k^{t+1}\right)\geq V_{t+2}^k\left(y_l^{t+1}\right)$, where we use the convention that $V_{\bar{t}+1}\equiv 0$. Hence, whenever $R_{t+1}^k\left(y_k^{t+1}\right)\neq 0$, we have $m^k\left(V_{t+2}\left(y_k^{t+1}\right);y_k^{t+1}\right)=V_{t+2}^k\left(y_k^{t+1}\right)+R_{t+1}^k\left(y_k^{t+1}\right)>V_{t+2}^k\left(y_k^{t+1}\right)\geq V_{t+2}^k\left(y_l^{t+1}\right)$, showing that $\hat{x}_{t+1}\left[y_k^{t+1}\right]\neq\hat{x}_{t+1}\left[y_l^{t+1}\right]$. This inequality holds for each component, as $\hat{x}_{t+1}^1+\hat{x}_{t+1}^2=D^1+D^2$. Since $p_t^k\left(y^t\right)\in(0,1)$ and u^j is strictly concave, we therefore have $u^j\left(\tilde{x}^j(\mathbf{y})\right)>p_t^k\left(y^t\right)u^i\left(\hat{x}_{t+1}^j\left[y_k^{t+1}\right](\bar{y})\right)+\left(1-p_t^k\left(y^t\right)\right)u^j\left(\hat{x}_{t+1}^j\left[y_k^{t+1}\right](\bar{y})\right)$ at each continuation $(\boldsymbol{\rho},\bar{y})$ of y^t. Thus, by (14), $E^j\left[u^j\left(\tilde{x}^j\right)|y^t\right]>p_t^k\left(y^t\right)E^j\left[u^i\left(\hat{x}_{t+1}^j\left[y_k^{t+1}\right]\right)|y^t\right]+\left(1-p_t^k\left(y^t\right)\right)E^j\left[u^j\left(\hat{x}_{t+1}^j\left[y_l^{t+1}\right]\right)|y^t\right]=V_{t+1}^j\left(y^t\right)$. ■

Proof. (*Theorem 3*) Suppose that there exists an equilibrium in which agents do not agree at some history y^0. Then, by Theorem 2, $V_1(y^0)$ is a Pareto-optimal utility level, thus $R_0^{\rho_0(y^0)}=0$. Thus, by the previous Lemma, for each $j,k\in N$, and each $s>0$, $P^j\left(R_s^k\left(y^s\right)1_{\{\rho_s=k\}}=0|y^0\right)=1$, yielding $E^j\left[R_s^k1_{\{\rho_s=k\}}|y^0\right]=0$. Therefore, by (5), $V_1(y^0)=(E^1[u^1(D^1)|y^0],E^2[u^2(D^2)|y^0])$, showing that D^1 is Pareto-optimal at y^0. Since D is independent of Y_0 it must also be Pareto-optimal at the beginning – a contradiction. ■

Proof. (*Theorem 4*) The proof is very similar to that of Proposition 3 in Yildiz (2001), and omitted here. ■

We will now prove Theorem 5. To this end, given any set $S\subset\mathbb{R}^2$, let us write $Hull[S]$ for the *convex hull* of S, the smallest covex set that includes S. Note that $Hull[U_0(\mathcal{P}_1\cup\mathcal{P}_2)]=\{\lambda U_0(p[1])+(1-\lambda)U_0(p[2])|\lambda\in[0,1],\ p[1]\in\mathcal{P}_1,p[2]\in\mathcal{P}_2\}$.

Lemma 5 $U_0(\mathcal{P})=Hull[U_0(\mathcal{P}_1\cup\mathcal{P}_2)]$.

Proof. Given any $p\in\mathcal{P}$, starting at history $\mathbf{h}_{0,1}$, the subgame of $G(C,p)$ that starts at some y^0 before the recognized agent makes his offer coincides with that of and $G(C,p[i_0])$, where $i_0=\rho_0\left(y^0\right)$ and $p[i_0]\in\mathcal{P}_{i_0}$ such that $p[i_0]_t^i=p_t^i$ for each $t>0$ and $i\in N$, while $p[i_0]_0^{i_0}=1$. Therefore, the utility level

for an agent i at equilibrium consumption can be expressed as a mixture of $U_0^i(p[1])$ and $U_0^i(p[2])$ with probabilities p_0^1 and $p_0^2 = 1 - p_0^1$. Therefore, $U_0^i(p) = p_0^1 U_0^i(p[1]) + (1-p_0^2)U_0^i(p[2]) \in Hull[U_0(\mathcal{P}_1 \cup \mathcal{P}_2)]$. On the other hand, given any $v = \lambda U_0(p[1]) + (1-\lambda)U_0(p[2]) \in Hull[U_0(\mathcal{P}_1 \cup \mathcal{P}_2)]$, we consider $p \in P$ such that $p_0^1 = \lambda$ and, at any y^s and any $t > 0$, $p_t^i(y^s) = p[i_0]_t^i(y^s)$ for each $i \in N$ where i_0 is the agent recognized at 0 according to y^s. Once again, $U_0(p) = p_0^1 U_0(p[1]) + (1-p_0^2)U_0(p[2])$, showing that $v = U_0(p) \in U_0(\mathcal{P})$. ■

Lemma 6 *$m^i(v; y^t)$ is a continuous function of v.*

Proof. Given any y^t and any $v \in \tilde{U}(X, y^t)$, $m^i(v; y^t) = \max_{\xi \in Dom(v)} \xi^i$, where $Dom(v) = \{\xi = (E^1[u^1 \circ x^1[y^t]|y^t], E_t^2[u^2 \circ x^2[y^t]]|y^t)|x[y^t] \in C_t[y^t], \xi \geq v\}$. Note that $Dom(v)$ is always non-empty and compact. Moreover, since u^i is continuous, $(v) \Rightarrow Dom(v)$ is clearly upper-semi continuous, and hence by the Maximum Theorem $m^i(v; y^t)$ is a continuous function of v. ■

Lemma 7 *U_0 is continuous in p.*

Proof. Using mathematical induction on t, we will show that $V_t[p](y^{t-1})$ is continuous in p at each t and y^{t-1}. Firstly, $V_{\bar{t}}^i[p](y^{\bar{t}-1}) = u^i(D^i(y^{\bar{t}-1}))$ is trivially continuous in p. Assume that $V_{t+1}[p](y^t)$ is continuous in p for each y^t for some $t < \bar{t}$. Since $m^i(\cdot; y^t)$ is continuous, it follows that $R_t^i(y^t) = m^i(V_{t+1}[p](y^t); y^t) - V_{t+1}^i[p](y^t)$ is continuous in p for each y^t and each $i \in N$. Thus,

$$\begin{aligned} V_t^i[p](y^{t-1}) &= E^i\left[R_t^i(y^t) 1_{\{\rho_t = i\}}|y^{t-1}\right] + E^i\left[V_{t+1}^i[p](y^t)|y^{t-1}\right] \\ &= p_t^i(y^{t-1}) E^i\left[R_t^i(y^t)|\rho_t = i, y^{t-1}\right] + E^i\left[V_{t+1}^i[p](y^t)|y^{t-1}\right] \end{aligned}$$

is continuous in p. ■

The payoff vectors from Pareto-optimal allocations form a connected curve PF in $\mathbb{R}^2$, which we call the Pareto-frontier. Given any two $\underline{v}, \bar{v} \in PF$, we write $[\underline{v}, \bar{v}]_{PF} = \{v \in PF | \min\{\underline{v}^1, \bar{v}^1\} \leq v^1 \leq \max\{\underline{v}^1, \bar{v}^1\}\}$ for the part of the Pareto-frontier connecting $\underline{v}$ to $\bar{v}$.

Since U_0 is continuous and $\mathcal{P}_i$ is connected, $U_0(\mathcal{P}_i)$ is also connected. But, by Theorem 2, we have $U_0(\mathcal{P}_i) \subseteq PF$. Thus,

$$U_0(\mathcal{P}_i) = [U_0(\underline{p}[i]), U_0(\bar{p}[i])]_{PF}. \tag{15}$$

Using Lemma 5, we obtain the following lemma.

Lemma 8 $U_0(\mathcal{P}) \cap PF = U_0(\mathcal{P}_1 \cup \mathcal{P}_2) = [U_0(\underline{p}[1]), U_0(\bar{p}[1])]_{PF} \cup [U_0(\underline{p}[1]), U_0(\bar{p}[1])]_{PF}$.

Proof. We have $U_0(\mathcal{P}) \cap PF = Hull[U_0(\mathcal{P}_1 \cup \mathcal{P}_2)] \cap PF = U_0(\mathcal{P}_1 \cup \mathcal{P}_2)$, where the first equality is due to Lemma 5, and the second equality is due to the fact that we have strictly concave utility functions yielding strictly convex set $U[X]$ of materially-feasible utility levels. By (15), we further have $U_0(\mathcal{P}_1 \cup \mathcal{P}_2) = [U_0(\underline{p}[1]), U_0(\bar{p}[1])]_{PF} \cup [U_0(\underline{p}[1]), U_0(\bar{p}[1])]_{PF}$. ■

Proof. (*Theorem 5*) (See Figure 2 for illustration.) Firstly, the set of all Pareto-optimal and individually rational utility levels is $[U_0(\bar{p}[1]), U_0(\bar{p}[2])]_{PF}$. Thus, Every Pareto-optimal and individually-rational allocation is obtainable iff $[U_0(\bar{p}[1]), U_0(\bar{p}[2])]_{PF} \subseteq U_0(\mathcal{P})$. This inclusion holds iff $U_0(\mathcal{P}_1) \cap U_0(\mathcal{P}_2) \neq \emptyset$. [If $U_0(\mathcal{P}_1) \cap U_0(\mathcal{P}_2) \neq \emptyset$, then by Lemma 8 we have $U_0(\mathcal{P}) \cap PF = U_0(\mathcal{P}_1) \cup U_0(\mathcal{P}_2) = [U_0(\bar{p}[1]), U_0(\bar{p}[2])]_{PF}$. Conversely, when $U_0(\mathcal{P}_1) \cap U_0(\mathcal{P}_2) = \emptyset$, by Lemma 8, $[U_0(\bar{p}[1]), U_0(\bar{p}[2])]_{PF} \backslash U_0(\mathcal{P}) = [U_0(\underline{p}[1]), U_0(\underline{p}[2])]_{PF} \neq \emptyset$.] But $U_0(\mathcal{P}_1) \cap U_0(\mathcal{P}_2) \neq \emptyset$ iff $U_0^1(\underline{p}[1]) \leq U_0^1(\underline{p}[2])$, and this inequality holds iff $U_0^2(\underline{p}[2]) \leq U_0^2(\underline{p}[1])$. ■

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